DIFFERENTIAL-SPACE

By Norbert Wiener

§1. **Introduction.** The notion of a function or a curve as an element in a space of an infinitude of dimensions is familiar to all mathematicians, and has been since the early work of Volterra on functions of lines. It is worthy of note, however, that the physicist is equally concerned with systems the dimensionality of which, if not infinite, is so large that it invites the use of limit-processes in which it is treated as infinite. These systems are the systems of statistical mechanics, and the fact that we treat their dimensionality as infinite is witnessed by our continual employment of such asymptotic formulae as that of Stirling or the Gaussian probability-distribution.

The physicist has often occasion to consider quantities which are of the nature of functions with arguments ranging over such a space of infinitely many dimensions. The density of a gas, or one of its velocity-components at a point, considered as depending on the coördinates and velocities of its molecules, are cases in point. He therefore is implicitly, if not explicitly, studying the theory of functionals. Moreover, he generally replaces any of these functionals by some kind of average value, which is essentially obtained by an integration in space of infinitely many dimensions.

Now, integration in infinitely many dimensions is a relatively little-studied problem. Apart from certain tentative investigations of Fréchet[1] and E. H. Moore[2], practically all that has been done on it is due to Gâteaux[3], Lévy[4], Daniell[5], and the author of this paper[6]. Of these investigations, perhaps the most complete are those begun by Gâteaux and carried out by Lévy in his *Leçons d'Analyse Fonctionnelle*. In this latter book, the mean value of the functional $U|[x(t)]|$ over the region of function-space

$$\int_0^1 [x(t)]^2 dt \leq 1$$

is considered to be the limit of the mean of the function.

$$U(x_1, \ldots, x_n) = U|[\xi_n(t)]|,$$

$$\left(\text{where} \qquad \xi_n(t) = x_k \quad \text{for} \quad \frac{k-1}{n} \leq t < \frac{k}{n}\right)$$

over the sphere

$$x_1^2 + x_2^2 + \ldots + x_n^2 = n$$

as n increases without limit.

The present paper owes its inception to a conversation which the author had with Professor Lévy in regard to the relation which the two systems of integration in infinitely many dimensions — that of Lévy and that of the author — bear to one another. For this indebtedness the author wishes to give full credit. He also wishes to state that a very considerable part of the substance of the paper has been presented, albeit from a different standpoint and employing different methods, in his previously

1 *Sur l'intégrale d'une fonctionnèlle étendue à un ensemble abstrait*, Bull. Soc. Math. de France, Vol. 43, pp. 249–267.

2 Cf. American Mathematical Monthly, Vol. 24 (1917), pp. 31, 333.

3 Two papers, published after his death by Lévy, Bulletin de la Société Mathématique de France, 1919.

4 P. Lévy, *Leçons d'Analyse Fonctionnelle*. Hereinafter to be referred to as " Lévy."

5 P. J. Daniell, *A General Form of Integral*, Annals of Mathematics, Series 2 Vol. 19, pp. 279–294. Hereinafter to be referred to as " Daniell." Also paper in Vol. 20.

6 N. Wiener, *The Average of an Analytic Functional*, Proc. Nat. Acad. of Sci., Vol. 7, pp. 253–26; *The Average of an Analytic Functional and the Brownian Movement*, *ibid*, 294–298. Also forthcoming paper in Proc. Lond. Math. Soc.

cited publications. It seemed better to repeat a little of what had already been done than to break the continuity of the paper by the constant reference to theorems in other journals and based on treatments so distinct from the present one that it would need a large amount of explanation to show their relevance.

§2. The Brownian Movement. When a suspension of small particles in a liquid is viewed under a microscope, the particles seem animated with a peculiar haphazard motion — the Brownian movement. This motion is of such an irregular nature that Perrin[7] says of it: " One realizes from such examples how near the mathematicians are to the truth in refusing, by a logical instinct, to admit the pretended geometrical demonstrations, which are regarded as experimental evidence for the existence of a tangent at each point of a curve." It hence becomes a matter of interest to the mathematician to discover what are the defining conditions and properties of these particle-paths.

The physical explanation of the Brownian movement is that it is due to the haphazard impulses given to the particles by the collisions of the molecules of the fluid in which the particles are suspended. Of course, by the laws of mechanics, to know the motion of a particle, one must know not only the impulses which it receives over a given time, but the initial velocity with which it is imbued. According, however, to the theory of Einstein,[8] this initial velocity over any ordinary interval of time, is of negligible importance in comparison with the impulses received during the time in question. Accordingly, the displacement of a particle during a given time may be regarded as independent of its entire previous history.

Let us then consider the time-equations of the path of a particle subject to the Brownian movement as of the form $x = x(t)$, $y = y(t)$, $z = z(t)$, t being the time and x, y, and z the coördinates of the particle. Let us limit our attention to the function $x(t)$. Since there is no appreciable carrying over of velocity from one instant to another, the difference between $x(t_1)$ and $x(t)$ $[t_1 > t]$ may be regarded as the sum of the displacements incurred by the particle

7 p. 64, *Brownian Movement and Molecular Reality*, tr. by F. Soddy.

8 Perrin, pp. 51–54.

over a set of intervals constituting the interval from t to t_1. In particular, if the constituent intervals are of equal size, then the probability-distribution of the displacements accrued in the different intervals will be the same. Since positive and negative displacements of the same size will, from physical considerations, be equally likely, it will be seen that for intervals of time large with comparison to the intervals between molecular collisions, by dividing them into many equal parts, which are still large with respect to the intervals between molecular collisions, and breaking up the total incurred displacement into the sum of displacements incurred in these intervals, we get very nearly a Gaussian distribution of our total displacement.[9] That is, the probability that $x(t_1)-x(t)$ lie between a and b is very nearly of the form

$$\frac{1}{\sqrt{\pi\phi(t_1-t)}}\int_a^b e^{-\frac{x^2}{\phi(t_1-t)}}\,dx. \tag{1}$$

Since the error incurred over the interval t to t_1 is the sum of the independent errors incurred over the periods from t to t_2 and from t_2 to t_1, we have

$$\begin{aligned}
&\frac{1}{\sqrt{\pi\phi(t_1-t)}}e^{-\frac{x^2}{\phi(t_1-t)}}\\
&=\frac{1}{\pi\sqrt{\phi(t_1-t_2)\phi(t_2-t)}}\int_{-\infty}^{\infty}e^{-\frac{y^2}{\phi(t_2-t)}-\frac{(x-y)^2}{\phi(t_1-t_2)}}dy\\
&=\frac{1}{\pi\sqrt{\phi(t_1-t_2)\phi(t_2-t)}}\int_{-\infty}^{\infty}exp\left\{-\left[y\sqrt{\frac{\phi(t_1-t_2)+\phi(t_2-t)}{\phi(t_1-t_2)\phi(t_2-t)}}\right.\right.\\
&\quad\left.\left.-\frac{x}{\sqrt{\phi(t_1-t_2)}}\sqrt{\frac{\phi(t_1-t_2)\phi(t_2-t)}{\phi(t_1-t_2)+\phi(t_2-t)}}\right]^2-\frac{x^2}{\phi(t_1-t_2)+\phi(t_2-t)}\right\}dy\\
&=\frac{e^{-\frac{x^2}{\phi(t_1-t_2)+\phi(t_2-t)}}}{\pi\sqrt{\phi(t_1-t_2)\phi(t_2-t)}}\int_{-\infty}^{\infty}e^{-y^2\left(\frac{\phi(t_1-t_2)+\phi(t_2-t)}{\phi(t_1-t_2)\phi(t_2-t)}\right)}dy\\
&=\frac{e^{-\frac{x^2}{\phi(t_1-t_2)+\phi(t_2-t)}}}{\sqrt{\pi\{\phi(t_1-t_2)+\phi(t_2-t)\}}}.
\end{aligned}\tag{2}$$

[9] Cf. Poincaré, *Le calcul des probabilités*, Ch. XI.

It is readily verified that this cannot be true unless

$$\phi(t_1-t)=\phi(t_1-t_2)+\phi(t_2-t), \tag{3}$$

whence

$$\phi(u)=Au, \tag{4}$$

and the probability that $x(t_1)-x(t)$ lie between a and b is of the form

$$\frac{1}{\sqrt{\pi A(t_1-t)}}\int_a^b e^{-\frac{x^2}{A(t_1-t)}}\,dx. \tag{5}$$

According to Einstein's theory,[10] $A=\dfrac{RT}{N}\dfrac{a}{3\pi a\xi}$ where R is the constant of a perfect gas, T is the absolute temperature, N is Avogadro's constant, a the radius of the spherical particles subject to the Brownian movement, and ξ the viscosity of the fluid containing the suspension.

§3. Differential-Space. In the Brownian movement, it is not the position of a particle at one time that is independent of the position of a particle at another; it is the displacement of a particle over one interval that is independent of the displacement of the particle over another interval. That is, instead of $f\left(\frac{1}{n}\right), \ldots, f\left(\frac{k}{n}\right), \ldots, f(1)$ representing "dimensions" of $f(t)$, the n quantities

$$\begin{aligned}
x_1 &= f\left(\frac{1}{n}\right)-f(0)\\
x_2 &= f\left(\frac{2}{n}\right)-f\left(\frac{1}{n}\right)\\
&\cdots\cdots\\
x_k &= f\left(\frac{k}{n}\right)-f\left(\frac{k-1}{n}\right)\\
&\cdots\cdots\\
x_n &= f(1)-f\left(\frac{n-1}{n}\right)
\end{aligned}$$

[10] Perrin, p. 53.

are of equal weight, vary independently, and in some degree represent dimensions. It is natural, then, to consider, as Lévy does, the properties of the sphere

$$x_1^2 + \ . \ . \ . + x_n^2 = r_n^2.$$

In particular, let us consider the measure, in terms of the whole sphere, of the region in which $f(a)-f(0)$, assuming it to be representable in terms of the x_s's, lies between the values α and β. Now, we have

$$f(a)-f(0) = \sum_1^{na} x_k. \tag{6}$$

Hence, by a change of coördinates, our question becomes: given that

$$\sum_1^n \xi_k^2 = r_n^2,$$

what is the chance that

$$\alpha \leq \sqrt{na}\, \xi_1 \leq \beta?$$

Letting $\xi_1 = \rho \sin \theta$, $\sqrt{\xi_2^2 + \ . \ . \ . + \xi_n^2} = \rho \cos \theta$, this chance becomes

$$\frac{\displaystyle\int_{\sin^{-1}\frac{\alpha}{r_n\sqrt{na}}}^{\sin^{-1}\frac{\beta}{r_n\sqrt{na}}} \cos^n\theta \, d\,\theta}{\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n \theta \, d\,\theta},$$

as has been indicated by Lévy[11]. This may also be written

$$\frac{\displaystyle\int_{\sqrt{n}\sin^{-1}\frac{\alpha}{r_n\sqrt{na}}}^{\sqrt{n}\sin^{-1}\frac{\beta}{r_n\sqrt{na}}} \cos^n \frac{x}{\sqrt{n}}\, dx}{\displaystyle\int_{-\frac{\sqrt{n}\pi}{2}}^{\frac{\sqrt{n}\pi}{2}} \cos^n \frac{x}{\sqrt{n}}\, dx}.$$

[11] Lévy, p. 266.

Now[12], for n large, $\cos^n \frac{x}{\sqrt{n}}$ converges uniformly to $e^{-\frac{x^2}{2}}$ for $|x| < A$. It may be deduced without difficulty from this fact that the integral in the denominator tends to $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$, which is $\sqrt{2\pi}$. As to the numerator, if the limits of integration approach limiting values, it will also converge. If in particular r_n is a constant, then since

$$\lim_{n\to\infty} \sqrt{n}\, \sin^{-1} \frac{\alpha}{r\sqrt{na}} = \frac{\alpha}{r\sqrt{a}},$$

while a like identity holds for the upper limit, we have for the probability (in the limit) that $f(a)$ lie between α and β.

$$\frac{1}{\sqrt{2\pi}} \int_{\frac{\alpha}{r\sqrt{a}}}^{\frac{\beta}{r\sqrt{a}}} e^{-\frac{x^2}{2}}\, dx = \frac{1}{r\sqrt{2\pi a}} \int_\alpha^\beta e^{-\frac{u^2}{2ar^2}}\, du. \tag{7}$$

Let it be noted that this expression is of exactly the form (5).

We shall call the space of which the constituent points are the functions $f(t)$, and in which the measure of a region is determined as the limit of a measure in u-space in the way in which formula (7) is obtained from (6), by the name *differential-space*. The appropriateness of this name comes from the fact that it is not the values of $f(t)$, but the small differences, that are uniformly distributed, and act as dimensions.

§4. The Non-Differentiability Coefficient of a Function. It thus appears that if we consider the distribution of $f(a)-f(0) = \sum_1^{na} x_k$ in the sphere $\sum_1^n x_k^2 = r^2$, we get in the limit a distribution of the values of $f(a)-f(0)$ essentially like the one indicated in (5). Now, we have

$$\sum_1^n x_k^2 = \sum_1^n \left\{ f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right\}^2.$$

[12] Lévy, p. 264.

Our result so far suggests, then, that the probability that $\sum_1^n \left\{ f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right\}^2$ exceed r^2 becomes increasingly negligible as n increases, or at least the probability that it exceed r^2 by a stated amount. On the hypothesis that $f(t_1)-f(t)$ have the distribution indicated in (5), and that the variation of f over one interval be independent of the variation of f over any preceding interval, let us discuss the distribution of

$$\sum_1^n \left\{ f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right\}^2.$$

Clearly, the chance that $\sum_1^n \left\{ f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right\}^2$ lie between α^2 and β^2 is the average value of a function of $x_1, \ldots, x_n$ which is 1 when $x_1^2 + \ldots + x_n^2$ lies between α^2 and β^2, and 0 otherwise, given that the weight of the region

$$\xi_k \leq x_k \leq \eta_k \qquad (k=1, 2, \ldots, n)$$

is

$$\prod_1^n \sqrt{\frac{n}{\pi A}} \int_{\xi_k}^{\eta_k} e^{-\frac{nx^2}{A}} dx$$

$$= \left(\sqrt{\frac{n}{\pi A}}\right)^n \int_{\xi_1}^{\eta_1} dx_1 \ldots \int_{\xi_n}^{\eta_n} dx_n \, e^{-\frac{n}{A}(x_1^2+\ldots+x_n^2)}, \qquad (9)$$

as follows from (5). Now, let us put

$$\left.\begin{aligned} x_1 &= \rho \sin\theta_1 \\ x_2 &= \rho \cos\theta_1 \sin\theta_2 \\ x_3 &= \rho \cos\theta_1 \cos\theta_2 \sin\theta_3 \\ &\ldots\ldots\ldots\ldots \\ x_{n-1} &= \rho \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{n-2} \sin\theta_{n-1} \\ x_n &= \rho \cos\theta_1 \cos\theta_2 \ldots \cos\theta_{n-1} \end{aligned}\right\}. \qquad (10)$$

We shall then have

$$\left.\begin{aligned} \rho^2 &= x_1^2 + \ldots + x_n^2 \\ dx_1 \ldots dx_n &= \rho^{n-1} \cos^{n-2}\theta_1 \cos^{n-3}\theta_2 \ldots \cos\theta_{n-2}\, d\rho\, d\theta_1 \ldots d\theta_{n-1} \end{aligned}\right\}, \qquad (11)$$

the last of which formulae may be readily demonstrated by an evaluation of the Jacobian

$$\begin{vmatrix} \frac{\partial x_1}{\partial \rho} & \frac{\partial x_1}{\partial \theta_1} & \ldots & \frac{\partial x_1}{\partial \theta_{n-1}} \\ \frac{\partial x_2}{\partial \rho} & \frac{\partial x_2}{\partial \theta_1} & \ldots & \frac{\partial x_2}{\partial \theta_{n-1}} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial x_n}{\partial \rho} & \frac{\partial x_n}{\partial \theta_1} & \ldots & \frac{\partial x_n}{\partial \theta_{n-1}} \end{vmatrix}. \qquad (12)$$

Employing a formula of Lévy to the effect that[13]

$$\int_0^{\frac{\pi}{2}} \cos^n\theta d\theta \int_0^{\frac{\pi}{2}} \cos^{n-1}\theta d\theta = \frac{\pi}{2n} \qquad (13)$$

we get for the chance that $\sum_1^n \left\{ f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right\}^2$ lie between α^2 and β^2

$$\left(\sqrt{\frac{n}{\pi A}}\right)^n \int_\alpha^\beta \rho^{n-1} e^{-\frac{n\rho^2}{A}} d\rho \prod_{k=1}^{n-2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^k\theta d\theta$$

$$= \left(\sqrt{\frac{n}{\pi A}}\right)^n 2^{n-2} \frac{\pi}{2(n-2)} \frac{\pi}{2(n-4)} \ldots \frac{\pi}{4} \int_\alpha^\beta \rho^{n-1} e^{-\frac{n\rho^2}{A}} d\rho$$

if n is even

$$= \left(\sqrt{\frac{n}{\pi A}}\right)^n 2^{n-2} \frac{\pi}{2(n-2)} \frac{\pi}{2(n-4)} \ldots \frac{\pi}{6} \int_\alpha^\beta \rho^{n-1} e^{-\frac{n\rho^2}{A}} d\rho$$

if n is odd. (14)

[13] Levy, p. 263.

The coefficients of the integral may be reduced in either case to the form

$$\frac{n^{\frac{n}{2}}}{\pi A^{\frac{n}{2}}\,\Gamma\left(\frac{n}{2}\right)},$$

so that the whole expression may be written, by a change of variable,

$$\frac{n^{\frac{n}{2}}}{\pi\Gamma\left(\frac{n}{2}\right)}\int_{\frac{\alpha}{\sqrt{A}}}^{\frac{\beta}{\sqrt{A}}} u^{n-1}\,e^{-nu^2}\,du. \tag{15}$$

The integrand vanishes for u zero and u infinite. Between those points it attains its maximum for $u^2=\frac{n-1}{2n}$, which for large values of n is in the neighborhood of 1/2. It hence becomes interesting to note how much is contributed to our integral by values of u^2 near 1/2 and how much by values remote from 1/2.

Let us then evaluate

$$\frac{n^{\frac{n}{2}}}{\pi\Gamma\left(\frac{n}{2}\right)}\int_{u^2=\frac{1}{2}+\epsilon}^{\infty} u^{n-1}\,e^{-nu^2}du,$$

remembering that the integrand is a decreasing function. Let us discuss the ratio

$$\frac{(u+1)^{n-1}\,e^{-n(u+1)^2}}{u^{n-1}\,e^{-nu^2}}=\left(1+\frac{1}{u}\right)^{n-1}e^{-n(2u+1)}$$

$$\leq\frac{3^{n-1}}{e^{2n}}<1/2. \tag{16}$$

We see at once that it follows from this that

$$\frac{n^{\frac{n}{2}}}{\pi\Gamma\left(\frac{n}{2}\right)}\int_{u^2=\frac{1}{2}+\epsilon}^{\infty} u^{n-1}\,e^{-nu^2}\,du<\frac{n^{\frac{n}{2}}}{\pi\Gamma\left(\frac{n}{2}\right)}\sum_{k=0}^{\infty}\frac{1}{2^k}\,(1/2+\epsilon)^{\frac{n-1}{2}}\,e^{-n(\frac{1}{2}+\epsilon)}$$

$$=\frac{2n^{\frac{n}{2}}}{\pi\Gamma\left(\frac{n}{2}\right)}\,(1/2+\epsilon)^{\frac{n-1}{2}}\,e^{-n(\frac{1}{2}+\epsilon)}. \tag{17}$$

For n large, by Stirling's theorem, this latter expression is asymptotically represented by

$$\frac{2n^{\frac{n}{2}}}{\pi\left(\frac{n}{2}-1\right)^{\frac{n}{2}-1}e^{1-\frac{n}{2}}\sqrt{2\pi\left(\frac{n}{2}-1\right)}}\left(\frac{1}{2}+\epsilon\right)^{\frac{n-1}{2}}e^{-n\left(\frac{1}{2}+\epsilon\right)}$$

$$=\frac{2}{\pi e}\sqrt{\frac{n-2}{\pi}}\,\frac{(1+2\epsilon)^{\frac{n-1}{2}}\,e^{-n\epsilon}}{\left(1-\frac{n}{2}\right)^{\frac{n}{2}}}. \tag{18}$$

As n grows larger, this in turn may be represented asymptotically by

$$\frac{2}{\pi(1+2\epsilon)^{\frac{1}{2}}}\sqrt{\frac{n-2}{\pi}}\left(\frac{1+2\epsilon}{e^{2\epsilon}}\right)^{\frac{n}{2}}. \tag{19}$$

Now, $e^{2\epsilon}>1+2\epsilon$, as may be seen directly from the Maclaurin series for e^x. Hence (19) may be written

$$K\sqrt{n-2}\,C^n,$$

where K is a constant and C is a positive constant less than 1. This, of course, tends to vanish for n large.

We may prove in a similar way that

$$\frac{n^{\frac{n}{2}}}{\pi\Gamma\left(\frac{n}{2}\right)}\int_0^{u^2=\frac{1}{2}-\epsilon} u^{n-1}\, e^{-nu^2}\, du$$

tends to vanish for n large. Clearly, for n sufficiently large, $\frac{n-1}{2n}$ will be greater than $1/2-\epsilon$. Under these circumstances,

$$\frac{n^{\frac{n}{2}}}{\pi\Gamma\left(\frac{n}{2}\right)}\int_0^{u^2=\frac{1}{2}-\epsilon} u^{n-1}\, e^{-nu^2}\, du \leq \frac{n^{\frac{n}{2}}}{2\pi\Gamma\left(\frac{n}{2}\right)}\left(\frac{1}{2}-\epsilon\right)^{\frac{n-1}{2}} e^{-n(\frac{1}{2}-\epsilon)}. \quad (20)$$

This latter quantity is asymptotically represented by

$$\frac{n^{\frac{n}{2}}}{2\pi\left(\frac{n}{2}-1\right)^{\frac{n}{2}-1} e^{1-\frac{n}{2}}\sqrt{2\pi\left(\frac{n}{2}-1\right)}}\left(\frac{1}{2}-\epsilon\right)^{\frac{n-1}{2}} e^{-n(\frac{1}{2}-\epsilon)}$$

$$=\frac{1}{2\pi\epsilon}\sqrt{\frac{n-2}{\pi}}\,\frac{(1-2\epsilon)^{\frac{n}{2}-1}\, e^{n\epsilon}}{\left(1-\frac{2}{n}\right)^{\frac{n}{2}}}. \quad (21)$$

This is in turn asymptotically represented by

$$\frac{1}{2\pi(1-2\epsilon)^{\frac{1}{2}}}\sqrt{\frac{n-2}{\pi}}\left[(1-2\epsilon)\, e^{2\epsilon}\right]^{\frac{n}{2}}. \quad (22)$$

Now, since by Taylor's theorem with remainder, $e^{-2\epsilon}=1-2\epsilon+\frac{\theta^2}{2}$, θ being a positive number less than 2ϵ, we get $(1-2\epsilon)e^{2\epsilon}<1$. Hence expression (22) is again of the form $K\sqrt{n-2}\;C^n$, C being a positive constant less than 1, and approaches zero as n increases.

It will be seen, then, that for n sufficiently large, the chance that $\sum_1^n\left\{f\left(\frac{k}{n}\right)-f\left(\frac{k-1}{n}\right)\right\}^2$ diverge from $A/2$ by more than ϵ is less than an expression of the form $K\sqrt{n-2}\;C^n$, in which K and C are positive constants independent of n, and $C<1$. It follows that for n sufficiently large, the chance that *any* $\sum\left\{f\left(\frac{k}{N}\right)-f\left(\frac{k-1}{N}\right)\right\}^2$ differ from $A/2$ by more than ϵ, for $N \geq n$, is less than

$$K\sum_n^\infty \sqrt{N-2}\; C^N,$$

which is the remainder of a convergent series, and hence vanishes as n becomes infinite. In other words, the chance that

$$\left|\lim_{n\to\infty}\sum_1^n f\left\{\left(\frac{k}{n}\right)-f\left(\frac{k-1}{n}\right)\right\}^2-A/2\right|>\epsilon$$

is less than any assignable positive number, if ϵ is any positive quantity.

Let $f(t)$ be a continuous function of limited total variation T between 0 and 1. Then clearly

$$\sum_1^n\left\{f\left(\frac{k}{n}\right)-f\left(\frac{k-1}{n}\right)\right\}^2\leq \max\left|f\left(\frac{k}{n}\right)-f\left(\frac{k-1}{n}\right)\right|\sum_1^n\left|f\left(\frac{k}{n}\right)\right.$$

$$\left.-f\left(\frac{k-1}{n}\right)\right|\leq T\max\left|f\left(\frac{k}{n}\right)-f\left(\frac{k-1}{n}\right)\right|. \quad (23)$$

Hence

$$\lim_{n\to\infty}\sum_1^n\left\{f\left(\frac{k}{n}\right)-f\left(\frac{k-1}{n}\right)^2\right\}=0. \quad (24)$$

This will in particular be the case when $f(t)$ possesses a derivative bounded over the closed interval (0, 1). Hence it is infinitely improbable, under our distribution of functions $f(t)$, that $f(t)$ be a continuous function of limited total variation, and in particular that it have a bounded derivative. We may regard $\lim_{n\to\infty}\sum_1^n\left\{f\left(\frac{k}{n}\right)-f\left(\frac{k-1}{n}\right)^2\right\}$ as in some sort a nondifferentiability coefficient of f.

§5. **The Maximum Gain in Coin-Tossing.** We have now investigated the differentiability of the functions $f(t)$; it behoves us to inquire as to their continuity. To this end we shall discuss the maximum value of $f(t)-f(t_0)$ incurred for $t_0 \leq t \leq t_1$, and shall determine its distribution. As a simple model of this rather complex situation, however, we shall consider a problem in coin-tossing.

A gambler stakes one dollar on each throw of a coin, which is tossed n times, losing the dollar if the throw is heads, and gaining it if the throw is tails. At the beginning he has lost nothing and won nothing; after m throws he will be k_m dollars ahead, k_m being positive or negative. The question here asked is, what is the distribution of the maximum value of k_m for $m \leq n$?

Let n throws be made in all the 2^n possible manners. Let $A_n(p)$ be the number of throw-sequences in which $\max(k_m)=p$. Clearly $A_n(-\mu)=0$, if μ is positive, since $k_0=0$. Furthermore, any throw-sequence in which the maximum gain is zero consists of a throw-sequence in which the maximum gain is either 1 or 0 preceded by a throw of heads. That is,

$$A_n(0)=A_{n-1}(0)+A_{n-1}(1). \tag{25}$$

A throw-sequence in which the maximum gain is $p>0$ consists either of a throw-sequence in which the maximum gain is $p-1$ preceded by a throw of tails, or a throw-sequence in which the maximum gain is $p+1$, preceded by a throw of heads. That is,

$$A_n(p)=A_{n-1}(p-1)+A_{n-1}(p+1). \tag{26}$$

Let us tabulate the first few values of $A_n(p)$. We get

$A_n(p)$	$n=1$	2	3	4	5	6
$p=0$	1	2	3	6	10	20
1	1	1	3	4	10	15
2	0	1	1	4	5	15
3	0	0	1	1	5	6
4	0	0	0	1	1	6
5	0	0	0	0	1	1
6	0	0	0	0	0	1

It will be seen that the various numbers in the table are the binomial coefficients, beginning in each column with the middle coefficient for n even and the next coefficient beyond the middle for n odd, and with every coefficient in the expansion $(1+1)^n$ repeated twice, with the exception of the middle coefficient for n even. That is, as far as the table is carried, we have

$$\left.\begin{aligned} A_{2n}(2p) &= A_{2n}(2p-1) = \frac{(2n)!}{(n+p)!\,(n-p)!} \\ A_{2n+1}(2p) &= A_{2n+1}(2p+1) = \frac{(2n+1)!}{(n+p+1)!\,(n-p)!} \end{aligned}\right\}. \tag{27}$$

We may then verify (25) and (26) by direct substitution, thus proving that the values given for $A_m(q)$ in (27) are valid for all values of m and q.

Let us now consider a series of n successive runs of m throws each, for points of $1/\sqrt{m}$ dollars, m being even. The chance that the amount gained in a single run lie between α and β dollars is then

$$1/2^m \sum_{\alpha'\sqrt{m}}^{\beta'\sqrt{m}} B_m(k), \tag{28}$$

where $\alpha'\sqrt{m}$ is the even integer next greater than or equal to $\alpha'\sqrt{m}$, $\beta'\sqrt{m}$ is the even integer next less than $\beta\sqrt{m}$ and $B_m(k)/2^m$ is the chance that of m throws of a coin just k more should be tails than heads — that is,

$$B_m(k)=\frac{m!}{\frac{m-k}{2}!\,\frac{m+k}{2}!}. \tag{29}$$

If we begin by assuming α and β positive and less than γ, expression (28) will lie between the values

$$\frac{(\beta'-\alpha')\sqrt{m}}{2^{m+1}} \frac{m!}{\left(\frac{m-\beta'\sqrt{m}}{2}\right)!\left(\frac{m+\beta'\sqrt{m}}{2}\right)!} \quad \text{and}$$

$$\frac{\beta('-\alpha')\sqrt{m}}{2^{m+1}} \frac{m!}{\left(\frac{m-\alpha'\sqrt{m}}{2}\right)!\left(\frac{m+\alpha'\sqrt{m}}{2}\right)!}.$$

Now, by Stirling's theorem, we have uniformly for $\alpha<\gamma$, $\beta<\gamma$

$$\lim_{m\to\infty}\frac{(\beta'-\alpha')\sqrt{m}}{2^{m+1}}\quad\frac{m!}{\left(\frac{m-\beta'\sqrt{m}}{2}\right)!\left(\frac{m+\beta'\sqrt{m}}{2}\right)!}$$

$$=\lim_{m\to\infty}\frac{(\beta-\alpha)\sqrt{m}\quad m^m e^{-m}\sqrt{2\pi m}}{2^{m+1}\left(\frac{m-\beta\sqrt{m}}{2}\right)^{\frac{m-\beta\sqrt{m}}{2}}\left(\frac{m+\beta\sqrt{m}}{2}\right)^{\frac{m+\beta\sqrt{m}}{2}}e^{-m}\pi\sqrt{m^2-\beta^2 m}}$$

$$=\lim_{m\to\infty}\frac{\beta-\alpha}{\sqrt{2\pi}}\left(\frac{m^2}{m^2-\beta^2 m}\right)^{\frac{m}{2}}\left(\frac{m-\beta\sqrt{m}}{m+\beta\sqrt{m}}\right)^{\frac{\beta\sqrt{m}}{2}}$$

$$=\lim_{m\to\infty}\frac{\beta-\alpha}{\sqrt{2\pi}}(1-\beta^2/m)^{\frac{m}{2}}(1-\beta/\sqrt{m})^{\frac{\beta\sqrt{m}}{2}}(1+\beta/\sqrt{m})^{-\frac{\beta\sqrt{m}}{2}}$$

$$=\frac{\beta-\alpha}{\sqrt{2\pi}}e^{-\frac{\beta^2}{2}}. \tag{30}$$

We have, then, for m sufficiently large, by combining (30) with an analogous theorem,

$$\frac{\beta-\alpha}{\sqrt{2\pi}}e^{-\frac{\beta^2}{2}}<1/2^m\sum_{\alpha'\sqrt{m}}^{\beta'\sqrt{m}}B_m(k)<\frac{\beta-\alpha}{\sqrt{2\pi}}e^{-\frac{\alpha^2}{2}}. \tag{31}$$

Moreover, since (30) holds uniformly, the value of m necessary to make (31) valid depends only on $\beta-\alpha$.

Now, let us divide the interval (α, β) into the p equal parts (α, α_1), (α_1, α_2), . . . , (α_{n-1}, β), and let us choose m sufficiently large for us to have over each interval

$$\frac{\alpha_{k+1}-\alpha_k}{\sqrt{2\pi}}e^{-\frac{\alpha^2_{k+1}}{2}}<1/2^m\sum_{\alpha'_k\sqrt{m}}^{\alpha''_{k+1}\sqrt{m}}B_m(k)<\frac{\alpha_{k+1}-\alpha_k}{\sqrt{2\pi}}e^{-\frac{\alpha^2_k}{2}},$$

where $\alpha_k'\sqrt{m}$ is $\alpha_k\sqrt{m}$ if this is an even integer, or otherwise the next larger integer, and $\alpha_k''\sqrt{m}=\alpha_k\sqrt{m}-2$. Then we have

$$\frac{1}{\sqrt{2\pi}}\sum_1^n(\alpha_{k+1}-\alpha_k)e^{-\frac{\alpha^2_{k+1}}{2}}<1/2^m\sum_{\alpha'\sqrt{m}}^{\beta'\sqrt{m}}B_m(k)$$

$$<\frac{1}{\sqrt{2\pi}}\sum_1^n(\alpha_{k+1}-\alpha_k)e^{-\frac{\alpha^2_k}{2}},$$

where the value of m needed depends only on $\alpha_{k+1}-\alpha_k$. Since $e^{-\frac{x^2}{2}}$ is a decreasing function, we hence have

$$\left|\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-\frac{x^2}{2}}dx-1/2^m\sum_{\alpha'\sqrt{m}}^{\beta'\sqrt{m}}B_m(k)\right|<\frac{\gamma}{\sqrt{2\pi}}\max\left(e^{-\frac{\alpha^2_k}{2}}-e^{-\frac{\alpha^2_{k+1}}{2}}\right)$$

$$<\frac{\gamma}{\sqrt{2\pi e}}(\alpha_{k+1}-\alpha_k), \tag{32}$$

where a minimum value of m may be determined in terms of $\alpha_{k+1}-\alpha_k$ alone. In other words, $1/2^m\sum_{\alpha'\sqrt{m}}^{\beta'\sqrt{m}}B_m(k)$ converges *uniformly* in α and β to

$$\frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-\frac{x^2}{2}}dx,$$

provided only α and β are positive and smaller than γ. It is obvious that the requirement of positiveness is superfluous, if only $|\alpha|<\gamma$, $|\beta|<\gamma$.

This last restriction can be removed. In the first place

$$B_m(2k)/B_m(k)=\frac{(m-k)!\,(m+k)!}{(m-2k)!\,(m+2k)!}$$

$$=\frac{(m-2k+1)(m-2k+2)\ .\ .\ .\ (m-k)}{(m+k+1)(m+k+2)\ .\ .\ .\ (m+2k)}$$

$$<\left(\frac{m-k}{m+2k}\right)^k. \tag{33}$$

If now $k>l\sqrt{m}$, we have

$$B_m(2k)/B_m(k)<\left(\frac{\sqrt{m}-l}{\sqrt{m}+2l}\right)^{\sqrt{m}}<\left(1-\frac{l}{\sqrt{m}}\right)^{\sqrt{m}}<l^{-l}, \tag{34}$$

for $k\leq m$. If $k>m$, we have clearly

$$B_m(2k)=B_m(k)=0.$$

Hence in general, for all $k > l\sqrt{m}$,

$$B_m(2^n k) \leq e^{-nl}\, B_m(k). \tag{34}$$

It follows that by making γ sufficiently large, we can make

$$\frac{1}{2^m}\sum_{\gamma'\sqrt{m}}^{\infty} B_m(k) \leq \frac{1}{2^m}\sum_{0}^{\gamma'\sqrt{m}} B_m(k)\ (1+e^{-l}+e^{-2l}+\ .\ .\ .)$$

$$\leq \frac{1}{2^m}\sum_{0}^{\gamma'\sqrt{m}} B_m(k)\ \frac{1}{1-e^{-l}}\ , \tag{35}$$

where $\gamma'\sqrt{m}$ is the integer next larger than $\gamma\sqrt{m}$ and l is independent of m. The result is that

$$\lim_{\gamma\to\infty} \frac{1}{2^m}\sum_{\gamma'\sqrt{m}}^{\infty} B_m(k) = 0 \tag{36}$$

uniformly in m. Hence if we consider

$$\left| \frac{1}{\sqrt{2\pi}} \int_{\alpha} e^{-\frac{x^2}{2}} dx - 1/2^{m+1} \sum_{\alpha'\sqrt{m}}^{\beta'\sqrt{m}} B_m(k) \right|, \tag{37}$$

we may first choose γ so large that the difference in this expression made by replacing α or β by $\pm\gamma$, in case they lie outside $(-\gamma, \gamma)$, is less than $\epsilon/2$, and then choose m so large that within the region $(-\gamma, \gamma)$, the expression is less than $\epsilon/2$. That is, by a choice of m alone, independently of α and β, we can make (37) less than ϵ.

Now let us write

$$\left.\begin{aligned} \frac{1}{2^{m+1}} \sum_{-\infty}^{\beta'\sqrt{m}} B_m(k) &= \phi_m(\beta) \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-\frac{x^2}{2}}\, dx &= \phi(\beta) \end{aligned}\right\}. \tag{38}$$

Clearly, the probability that the maximum amount won for any value of k at the end of the first km throws out of mn for $\sqrt{m}$

dollars each lie between 0 and u will be the Stieltjes integral expression

$$\int_{0-0}^{u} d\phi_m(x_1) \int_{-\infty}^{-x_1} d\phi_m(x_2) \int_{-\infty}^{-x_1-x_2} d\phi_m(x_3) \ldots \int_{-\infty}^{-x_1-x_2-\cdots-x_{n-1}} d\phi_m(x_n)$$

$$+\int_{0+0}^{u} d\phi_m(x_1) \int_{0+0}^{u-x_1} d\phi_m(x_2) \int_{-\infty}^{-x_1-x_2} d\phi_m(x_3) \ldots \int_{-\infty}^{-x_1-x_2-\cdots-x_{n-1}} d\phi_m(x_n)$$

$$+\int_{0+0}^{u} d\phi_m(x_1) \int_{-\infty}^{0} d\phi_m(x_2) \int_{x_1+0}^{u} d\phi_m(x_3) \ldots \int_{-\infty}^{-x_1-x_2-\cdots-x_{n-1}} d\phi_m(x_n)$$

$$+\int_{0+0}^{u} d\phi_m(x_1) \int_{0+0}^{u-x_1} d\phi_m(x_2) \int_{x_1+x_2+0}^{u} d\phi_m(x_3) \ldots \int_{-\infty}^{-x_1-x_2-\cdots-x_{n-1}} d\phi_m(x_n)$$

$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\int_{0+0}^{u} d\phi_m(x_1) \int_{0+0}^{u-x_1} d\phi_m(x_2) \ \ldots\ \int_{0+0}^{u-x_1-x_2-\cdots-x_{n-1}} d\phi_m(x_n), \tag{39}$$

consisting of $1+1+2!+3!+\ .\ .\ +n!$ terms. If in this expression we substitute ϕ for ϕ_m, it results from what we have just said of the uniformity of the convergence of ϕ_m to ϕ that the expression we obtain will be uniformly the limit of (39). This expression will represent the chance that after n independent games, in each of which the chance of winning a sum between α dollars and β dollars is $\frac{1}{\sqrt{2\pi}}\int_{\alpha}^{\beta} e^{-\frac{x^2}{2}}\, dx$, the maximum gain incurred lies between 0 and u. We shall denominate this quantity $G(u, n)$, and expression (39) $G_m(u, n)$.

It is obvious from the definition that $G(u, n)$ is more than

$$1/2^{mn}\sum_{0}^{u'\sqrt{m}} A_{mn}(k) = 1/2^{mn-1} \sum_{k=0}^{k=u'\sqrt{m}} \frac{mn!}{\left(\frac{mn-k}{2}\right)!\ \left(\frac{mn+k}{2}\right)!}\ , \tag{40}$$

where $u'\sqrt{m}$ is the integer next larger than $u\sqrt{m}$. By going through an argument precisely analogous to that by which (32) was obtained, we see that

$$\lim_{m\to\infty} 1/2^{mn}\sum_{0}^{u'\sqrt{mn}} A_{mn}(k) = \sqrt{\frac{2}{\pi}}\int_0^{u\sqrt{n}} e^{-\frac{x^2}{2}}\,dx$$

$$= \sqrt{\frac{2}{n\pi}}\int_0^{u} e^{-\frac{x^2}{2n}}\,dx. \tag{41}$$

Hence

$$G\,(u,n) > \sqrt{\frac{2}{n\pi}}\int_0^{u} e^{-\frac{x^2}{2n}}\,dx. \tag{42}$$

§6. **Measure in Differential-Space.** To revert to differential-space, let us consider a functional $F|f|$, f being defined for arguments between zero and one, and let us see if we can frame a definition of its average. To begin with, let us suppose that F only depends on the values of f for the arguments $t_1 < t_2, \ldots < t_n$. F will then be unchanged if we alter f to any other function, assuming the same values for these arguments — in particular, to a step-function $f_\nu(t)$ with steps all of length $1/\nu$, ν being sufficiently large.

Let us now call the difference between the height of the kth step and that of the $(k-1)$st by the name x_k. We shall then have

$$\begin{aligned} F|f_\nu| &= F(f_\nu(t_1), f_\nu(t_2), \ldots, f_\nu(t_n)) \\ &= F(x_1+x_2+\ldots+x_{T_1}, x_1+x_2+\ldots+x_{T_2}, \ldots, \\ &\qquad x_1+x_2+\ldots+x_{T_n}), \end{aligned} \tag{43}$$

where T_k is νt_k if this is an integer, and otherwise the next smaller integer. Now, the region of the space $(x_1, x_2, \ldots, x_n)$ which corresponds to differential-space is the interior of the sphere

$$x_1^2 + x_2^2 + \ldots + x_n^2 = r^2.$$

Over the interior of this sphere we may take the average of expression (43), let it approach a limit as ν increases indefinitely, and call this limit the average of F in differential-space.

By the change in coördinates

$$\left.\begin{aligned} \ldots+x_2 + \ldots + x_{T_1} &= \sqrt{T_1}\,\xi_1 \\ \ldots_{T_1+1} + \ldots + x_{T_2} &= \sqrt{T_2-T_1}\,\xi_2 \\ \ldots\quad\ldots\quad&\ldots \\ \ldots_{T_{n-1}+1} + \ldots + x_{T_n} &= \sqrt{T_n-T_{n-1}}\,\xi_n, \\ \xi_{n+1}, \ldots, \xi_n \text{ represent coördinates orthogonal to } &\xi_1, \ldots, \xi_n \end{aligned}\right\} \tag{44}$$

...e change our sphere into

$$\xi_1^2 + \ldots + \xi_\nu^2 = r^2,$$

...nd have also

$$F|f_\nu| = F(\sqrt{T_1}\xi_1, \sqrt{T_2-T_1}\,\xi_2, \ldots, \sqrt{T_n-T_{n-1}}\,\xi_n).$$

By a transformation like (10), this will become

$$F|f_\nu| = F(\sqrt{T_1}\,\rho\sin\theta_1, \sqrt{T_2-T_1}\,\rho\cos\theta_1\sin\theta_2, \ldots \sqrt{T_n-T_{n-1}}\,\rho\cos\theta_1, \ldots \cos\theta_{n-1}\sin\theta_n).$$

The average value of this over our sphere is

$$\frac{\left\{\begin{aligned}&\int_0^r d\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_1\ldots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_{\nu-1}\,\rho^{\nu-1}\cos^{\nu-2}\theta_1\cos^{\nu-3}\theta_2\ldots\cos\theta_{\nu-2}\\ &F(\sqrt{T_1}\,\rho\sin\theta_1,\ldots,\sqrt{T_n-T_{n-1}}\,\rho\cos\theta_1\ldots\sin\theta_n)\end{aligned}\right\}}{\int_0^r d\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_1,\ldots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_{\nu-1}\,\rho^{\nu-1}\cos^{\nu-2}\theta_1\ldots\cos\theta_{\nu-2}}$$

$$= \frac{\left\{\begin{aligned}&\int_0^r d\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_1\ldots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_n\rho^{\nu-1}\cos^{\nu-2}\theta_1\ldots\cos^{\nu-n-1}\theta_n\\ &F\,(\sqrt{T_1}\,\rho\,\sin\theta_1,\ldots,\sqrt{T-T_{n-1}}\,\rho\cos\theta_1\ldots\sin\theta_n)\end{aligned}\right\}}{\int_0^r d\rho\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_1\ldots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta_n\,\rho^{\nu-1}\cos^{\nu-2}\theta_1\ldots\cos^{\nu-n-1}\theta_n}$$

$$= \frac{\left\{ \begin{aligned} &\int_0^r d\rho \int_{-\frac{\pi\sqrt{\nu}}{2}}^{\frac{\pi\sqrt{\nu}}{2}} du_1 \cdots \int_{-\frac{\pi\sqrt{\nu}}{2}}^{\frac{\pi\sqrt{\nu}}{2}} du_n \rho^{\nu-1} \cos^{\nu-2}\frac{u_1}{\sqrt{\nu}} \cdots \cos^{\nu-n-1}\frac{u_n}{\sqrt{\nu}} \\ &F\left(\sqrt{T}\rho \sin\frac{u_1}{\sqrt{\nu}}, \ldots, \sqrt{T_n - T_{n-1}}\,\rho \cos\frac{u_1}{\sqrt{\nu}}, \ldots \sin\frac{u_n}{\sqrt{\nu}}\right) \end{aligned} \right\}}{\int_0^r d\rho \int_{-\frac{\pi\sqrt{\nu}}{2}}^{\frac{\pi\sqrt{\nu}}{2}} du_1 \cdots \int_{-\frac{\pi\sqrt{\nu}}{2}}^{\frac{\pi\sqrt{\nu}}{2}} du_n \rho^{\nu-1} \cos^{\nu-2}\frac{u_1}{\sqrt{\nu}} \cdots \cos^{\nu-n-1}\frac{u_n}{\sqrt{\nu}}} \tag{45}$$

If we now consider the integrands in the numerator and the denominator for $|u_1| < U, \ldots, |u_n| < U$, and let ν increase indefinitely, we see[14] that

$$\lim_{\nu\to\infty} \cos^{\nu-2}\frac{u_1}{\sqrt{\nu}} \cdots \cos^{\nu-n-1}\frac{u_n}{\sqrt{\nu}} F\left(\sqrt{T_1}\rho\sin\frac{u_1}{\sqrt{\nu}}, \ldots, \sqrt{T_n - T_{n-1}}\,\rho\cos\frac{u_1}{\sqrt{\nu}} \cdots \sin\frac{u_n}{\sqrt{\nu}}\right)$$

$$= e^{-\frac{u_1^2}{2} - \cdots - \frac{u_n^2}{2}} F(\rho u_1\sqrt{t_1}, \ldots, \rho u_n\sqrt{t_n - t_{n-1}}), \tag{46}$$

and

$$\lim_{\nu\to\infty} \cos^{\nu-2}\frac{u_1}{\sqrt{\nu}} \cdots \cos^{\nu-n-1}\frac{u_n}{\sqrt{\nu}} = e^{-\frac{u_1^2}{2} - \cdots - \frac{u_n^2}{2}} \tag{47}$$

uniformly, provided only F is continuous. If in addition F vanishes for $|u_1| > U, \ldots, |U_n| > U$, we have for the limit of the expression in (45)

$$\lim_{\nu\to\infty} \frac{\left\{ \begin{aligned} &\int_0^r \rho^{\nu-1} d\rho \int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n\, e^{-\frac{u_1^2}{2} - \cdots - \frac{u_n^2}{2}} \\ &F(\rho u_1\sqrt{t_1}, \ldots, \rho u_n\sqrt{t_n - t_{n-1}}) \end{aligned} \right\}}{\int_0^r \rho^{\nu-1} d\rho \int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n\, e^{-\frac{u_1^2}{2} - \cdots - \frac{u_n^2}{2}}}.$$

[14] Cf. note 12.

This will be the case even if F has a set of discontinuities of zero measure, provided F vanishes for $|u_1| > U, \ldots, |U_n| > U$, or even, as may be shown without difficulty by a limit argument, provided merely that $|F|$ is bounded. It follows at once that, under these conditions, the expression in (45) becomes

$$\frac{\int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n\, e^{-\frac{u_1^2}{2} - \cdots - \frac{u_n^2}{2}} F(ru_1\sqrt{t_1}, \ldots, ru_n\sqrt{t_u - t_{n-1}})}{\int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n\, e^{-\frac{u_1^2}{2} - \cdots - \frac{u_n^2}{2}}}$$

$$= (2\pi)^{-\frac{n}{2}} \int_{-\infty}^{\infty} du_1 \cdots \int_{-\infty}^{\infty} du_n\, e^{-\frac{u_1^2}{2} - \cdots - \frac{u_n^2}{2}} F(ru_1\sqrt{t_1}, \ldots, ru_n\sqrt{t_n - t_{n-1}})$$

$$= (2\pi)^{-\frac{n}{2}} r^{-n} [t_1(t_2 - t_1) \cdots (t_n - t_{n-1})]^{-\frac{1}{2}} \int_{\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n \; exp \sum_{1}^{n} \frac{y_k^2}{2r^2(t_k - t_{k-1})} F(y_1, \ldots y_n), \tag{48}$$

to being zero. In particular, if $F(y_1, \ldots, y_n)$ is 1 when

$$\left. \begin{aligned} y_{11} &< y_1 < y_{12} \\ y_{21} &< y_2 < y_{22} \\ &\cdots\cdots \\ y_{n1} &< y_n < y_{n2} \end{aligned} \right\}, \tag{49}$$

we get for the average value of F

$$(2\pi)^{-\frac{n}{2}} r^{-n} [t_1 (t_2 - t_1) \cdots (t_n - t_{n-1})]^{-\frac{1}{2}} \int_{y_{11}}^{y_{12}} dy_1 \cdots \int_{y_{n1}}^{y_{n2}} dy_n \; exp \sum_{1}^{n} \frac{y_k^2}{2r^2(t_k - t_{k-1})}. \tag{50}$$

This we shall term the *measure* or *probability* of region (49).

§7. **Measure and Equal Continuity.** Let us now consider all those functions $f(t)$ $(0 \leq t \leq 1)$ for which $f(0)=0$ and which for some pair of rational arguments, $0 \leq t_1 < t_2 \leq 1$, satisfy the inequality

$$|f(t_2)-f(t_1)| > ar(t_2-t_1)^{\frac{1}{2}-\epsilon}. \tag{51}$$

If this inequality is satisfied, we can certainly find an n such that

$$\frac{k-1}{n} \leq t_1 \leq \frac{k}{n} \leq t_2 \leq \frac{k+1}{n}, \quad t_2-t_1 \geq 1/n, \tag{52}$$

while the variation of f over the interval $\left(\frac{k-1}{n}, \frac{k+1}{n}\right)$ will exceed $ar(t_2-t_1)^{\frac{1}{2}-\epsilon}$. That is, the class of functions satisfying the inequality (51) is a sub-class of the class of functions for which the variation over some interval $\left(\frac{k-1}{n}, \frac{k}{n}\right)$ exceeds $\frac{ar}{2} n^{\epsilon-\frac{1}{2}}$. Now, every such function must have either the difference between some *positive* value in the interval and its initial value exceed $\frac{ar}{4} n^{\epsilon-\frac{1}{2}}$, or the difference between its initial value and some *negative* value in the interval must exceed $\frac{ar}{4} n^{\epsilon-\frac{1}{2}}$.

To discuss what we may interpret as the measure of the set of functions (51), we may hence investigate the measure of the set of functions which over an interval of length $1/n$ depart from their initial value (for some rational argument, be it understood) by more than $\frac{ar}{4} n^{\epsilon-\frac{1}{2}}$. Now, if we subdivide the interval of length $1/n$ into m parts, each of length $1/mn$, the probability that in a given one of these sub-intervals the total difference between the initial and the final value of the function lie between α and β is, by (50)

$$\frac{\sqrt{mn}}{r\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-\frac{mnx^2}{2r^2}}\, dx. \tag{53}$$

Hence the probability that the maximum difference between the value of f at the end of one of these sub-intervals and its initial value be greater than $\frac{ar}{4} n^{\epsilon-\frac{1}{2}}$ is by (42) less than

$$\frac{1}{r}\sqrt{\frac{2n}{\pi}} \int_{\frac{ar}{4}n^{\epsilon-1/2}}^{\infty} e^{-\frac{nx^2}{2r^2}}\, dx$$

$$< \frac{1}{r}\sqrt{\frac{2n}{\pi}} \int_{\frac{ar}{4}n^{\epsilon-1/2}}^{\infty} e^{-\sqrt{\frac{nx^2}{2r^2}}}\, dx \left(\text{for } \frac{n^{2\epsilon}a^2r^2}{16} \geq 1\right)$$

$$= \frac{1}{r}\sqrt{\frac{2n}{\pi}}\sqrt{\frac{n}{2r^2}}\, e^{-\frac{n^\epsilon ar}{4}}$$

$$= \frac{n}{r^2\sqrt{\pi}}\, e^{-\frac{n^\epsilon ar}{4}}. \tag{54}$$

This will be true no matter how large m is. Hence those functions for which the maximum difference between $f\left(\frac{k}{n}+\frac{l}{mn}\right)$ and $f\left(\frac{k}{n}\right)$ is greater than $\frac{ar}{4} n^{\epsilon-1/2}$, l being an integer not greater than m, which is any integer whatever, can be included in a denumerable set of regions such as those contemplated in the last section, of total measure less than $\frac{n}{r^2\sqrt{\pi}} e^{-\frac{n^\epsilon ar}{4}}$. If now we let k vary from 1 to n, we shall find that those functions whose positive excursion in one of the intervals $\left(\frac{k}{n}, \frac{k+1}{n}\right)$ for some rational argument exceeds $\frac{ar}{4} n^{\epsilon-1/2}$ will have a total measure less than $\frac{n^2}{r^2\sqrt{\pi}} e^{-\frac{n^\epsilon ar}{4}}$, in the sense that they can be included in a denumerable assemblage of sets measurable in accordance with the provisions of the last section, and of total measure less than this amount. If in this last sentence we replace the words "positive excursion" by the words "positive or negative excursion," we get for the total measure of our set of functions a quantity less than $\frac{2n^2}{r^2\sqrt{\pi}}\, e^{\frac{n^\epsilon ar}{4}}$. It follows that all the functions satisfying (51) can be included in a denumerable set of regions of total measure less than

$$\frac{2}{r^2\sqrt{\pi}}\sum_{1}^{\infty} n^2\, e^{-\frac{n^\epsilon ar}{4}}. \tag{55}$$

This series converges if a has a sufficiently large value independent of n, as may be seen by comparing it with the series.

$$\frac{2}{r^2\sqrt{\pi}}\sum_{1}^{\infty} n^2\, (n^\epsilon)^{-e^{\frac{ar}{4}}},$$

which converges for a sufficiently large. Moreover, if (55) converges for $a=a_1$, we may write (55) in the form

$$\frac{2}{r^2\sqrt{\pi}}\sum_{1}^{\infty} n^2\, e^{-\frac{n^\epsilon ar}{4}} = \frac{2}{r^2\sqrt{\pi}}\sum_{1}^{\infty} n^2\, e^{-\frac{n^\epsilon a_1 r}{4}}\, e^{\frac{n^\epsilon (a_1-a)r}{4}}$$

$$< \left[\frac{2}{r^2\sqrt{\pi}}\sum_{1}^{\infty} n^2\, e^{-\frac{n^\epsilon a_1 r}{4}}\right] e^{\frac{(a_1-a)r}{4}} \tag{56}$$

for $a>a_1$. Hence we have

$$\lim_{a\to\infty}\frac{2}{r^2\sqrt{\pi}}\sum_{1}^{\infty} n^2\, e^{-\frac{n^\epsilon ar}{4}} = 0. \tag{57}$$

That is, by making a in (51) sufficiently large, the functions satisfying (51) may be included in a denumerable set of such regions as those discussed in §6, of total measure as small as may be desired.

Let us now consider the sphere

$$x_1^2 + \ldots + x_n^2 = r^2,$$

and on this sphere, a sector subtended at the center by a given area of the surface. Let us cut off a portion of this sector by a concentric sphere of radius $r_1 < r$, and let us measure the volume of this new sector in terms of the sphere of radius r. Let us compare the quantity thus obtained with

$$(2\pi)^{-\frac{n}{2}} r^{-n} n^{\frac{n}{2}} \int dx_1 \ldots \int dx_n\, e^{-\frac{(x_1^2+\ldots+x_n^2)n}{2r^2}}, \tag{58}$$

e integral being taken over the same sector. The ratio between ese two quantities may readily be shown to be

$$\frac{\dfrac{r^{-n}\left(\frac{n}{2}\right)!}{\pi^{\frac{n}{2}}}\displaystyle\int_0^{r_1}\rho^{n-1}\,d\rho}{(2\pi)^{-\frac{n}{2}} r^{-n} n^{\frac{n}{2}}\displaystyle\int_0^{r_1}\rho^{n-1} e^{-\frac{\rho^2 n}{2r^2}}\,d\rho}$$

$$= \frac{\left(\frac{n}{2}\right)!\displaystyle\int_0^{\frac{r_1\sqrt{n}}{r}} u^{n-1}du}{\left(\frac{n}{2}\right)^{\frac{n}{2}}\displaystyle\int_0^{\frac{r_1\sqrt{n}}{r}} u^{n-1} e^{-\frac{u^2}{2}}\,du}. \tag{59}$$

ince $e^{-\frac{u^2}{2}}$ is a decreasing function, we have

$$\frac{\left(\frac{n}{2}\right)!\displaystyle\int_0^{\frac{r_1\sqrt{n}}{r}} u^{n-1}\,du}{\left(\frac{n}{2}\right)^{\frac{n}{2}}\displaystyle\int_0^{\frac{r_1\sqrt{n}}{r}} u^{n-1} e^{-\frac{u^2}{2}}\,du}$$

$$> \frac{\left(\frac{n}{2}\right)!\displaystyle\int_0^{\sqrt{n}} u^{n-1}du}{\left(\frac{n}{2}\right)^{\frac{n}{2}}\displaystyle\int_0^{\sqrt{n}} u^{n-1} e^{-\frac{u^2}{2}}\,du}$$

$$> \frac{\left(\frac{n}{2}\right)!\, n^{\frac{n}{2}-1}}{\left(\frac{n}{2}\right)^{\frac{n}{2}}\displaystyle\int_0^{\infty} u^{n-1} e^{-\frac{u^2}{2}}\,du}$$

$$= \frac{\left(\frac{n}{2}\right)!\, n^{\frac{n}{2}-1}}{\left(\frac{n}{2}\right)^{\frac{n}{2}}\left(\frac{n}{2}-1\right)!\, 2^{\frac{n}{2}-1}} = 1. \tag{60}$$

Hence if we consider a region of the sphere $\Sigma x_k^2 = r^2$ bounded by radii and the sphere $\Sigma x_k^2 = r_1^2$, its measure in terms of the volume of the sphere $\Sigma x_k^2 = r^2$ is greater than its measure as given by an expression such as (58). In this statement we may replace the sphere $\Sigma x_k^2 = r_1^2$ by any region such that if it contains a point $(x_1, \ldots, x_n)$ it contains $(\theta x_1, \ldots, \theta x_n)$ $(0 \leq \theta \leq 1)$, since such a region may be approximated to within any desired degree of accuracy by a sum of sectors. It hence results that the measure of all the points in $\Sigma x_k^2 = r^2$ but *without* such a region is less than the measure after the fashion of (58) of all the points without such a region. We may conclude from this fact and (57) that in the space $\Sigma x_k^2 = r^2$, the measure of all of the points $(x_1, \ldots, x_n)$ such that for some k and $l > k$,

$$\left| \sum_{k+1}^{l} x_i \right| > a\, r\left(\frac{l-k}{n}\right)^{\frac{1}{2}-\epsilon} \tag{61}$$

vanishes uniformly in n as a increases.

§8. **The average of a Bounded, Uniformly Continuous Functional.** Let $F\,|f|$ be a functional, defined in the first instance for all step-functions constant over each of the intervals $\left(\frac{k}{n}, \frac{k+1}{n}\right)$ for some n, and having the following properties:

(1) $F|f| < A$ for all f,

(2) There is a function $\phi(x)$ such that

$$\lim_{x \to 0} \phi(x) = 0,$$

and

$$|\, F\,|f| - F\,|f+g|\, | < \phi(\max |g|).$$

We shall speak of the function $F(x_1, \ldots, x_n)$, which is $F\,|f_n|$, where

$$f_n(t) = \sum_{1}^{k} x_i \quad \text{for} \quad \frac{k-1}{n} < t \leq \frac{k}{n},$$

as the *nth section* of $F\,|f|$. I say that for any functional $F\,|f|$

satisfying (1) and (2), the average of the nth section over the sphere $\sum_{1}^{n} x_k^2 = r^2$ approaches a limit as n increases indefinitely, which limit we shall term the *average of* $F\,|f|$.

To begin with, let us have given a positive number η; we shall construct a value of n such that the difference between the averages of the nth and the $(n+p)$th sections of F is always less than 2η. First choose a so great that the measure of the points in $\Sigma x_k^2 \leq r^2$ satisfying (61) is for all n less than $\eta/2A$. Next find a number ξ so small that

$$\phi(ar\xi^{1/2-\epsilon}) < \eta/4. \tag{62}$$

Divide the interval (0, 1) into portions of width no greater than ξ, and let $t_1, \ldots, t_m$ be the boundaries of these intervals. Let

$$\psi(t) = f(t_n) \qquad (t_n \leq t \leq t_{n+1}),$$

and let

$$G|f| = F|\psi|.$$

Form the average of G as in §6, and let n be so large that this average differs from the average of

$$G(x_1 + x_2 + \ldots + x_{T_1}, x_1 + x_2 + \ldots + x_{T_2}, \ldots, x_1 + x_2 + \ldots + x_{T_m})^{15}$$

by less than $\eta/4$. Then it is immediately obvious that the difference between the averages of the nth and the $(n+p)$th sections of F is less than 2ϵ. In other words, the average of a functional satisfying (1) and (2) necessarily exists.

An example of such a functional is

$$\frac{1}{1 + \int_0^1 [f(x)]^2 dx}.$$

Another method of defining the average of a bounded, uniformly continuous functional F is as

$$\lim_{n \to \infty} (2\pi)^{-\frac{n}{2}} r^{-n} n^{\frac{n}{2}} \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n\, e^{-\frac{(x_1^2 + \ldots + x_n^2)n}{2r^2}} F(x_1, \ldots, x_n). \tag{63}$$

[15] Cf. (43).

There is no particular difficulty in showing the equivalence of the two definitions with the aid of (57). A fuller discussion of this definition of the average of a functional is to be found in the papers of the author which have been already cited. It is easy to demonstrate, as the author has done, that the division into parts that are exactly equal plays no essential rôle in definition (63), and can be replaced by a much more general type of division.

For the average of $F\,|f|$ as here defined, we shall write

$$A_r\{F\}. \tag{64}$$

§9. **The Average of an Analytic Functional.** Let $F|f|$ be a functional such that:

(1) Given any positive number B, there is an increasing function $A(B)$, such that if

$$\max |f(t)| \leq B,$$

then

$$|F|f|| < A(B).$$

(2) Given any positive number B, there is a function $\phi(x)$ such that

$$\lim_{x\to 0} \phi(x) = 0,$$

while if

$$\max |f(t)| \leq B, \quad \max |g(t)| \leq B$$

then

$$|F|f| - F|g|| < \phi(\max |f-g|).$$

Let us write

$$\left.\begin{aligned} F_H|f| &= F|f| && \text{for } \max |f(t)| \leq H \\ F_H|f| &= F\left|\frac{f(H)}{\max |f(t)|}\right| && \text{for } \max |f(t)| > H \end{aligned}\right\} \tag{65}$$

$F_H|f|$ is clearly bounded and uniformly continuous, and as such comes under the class of those functionals which have averages in the sense of the last paragraph.

Let us define for any functional $G|f|$ the function $G(x_1, \ldots, x_n)$ as in the last paragraph, and let us write $A_r^n\{F\}$ for the average of $F\,(x_1, \ldots, x_k)$ over $\Sigma x_k^2 = r^2$. Let H and $K > H$ be any two positive numbers. Clearly by (42) and the argument which follows (60), the set of functions for which $\max |f(t)| \geq H$ can be enclosed over $\Sigma x_k^2 = r^2$ in a region of measure

$$\frac{2}{r}\sqrt{\frac{2}{\pi}}\int_H^\infty e^{-\frac{x^2}{2r^2}}\,dx.$$

Hence

$$|A_r^n\{F_H\} - A_r^n\{F_K\}| < \frac{4}{r}\sqrt{\frac{2}{\pi}}\int_H^\infty e^{-\frac{x^2}{2r^2}}\,dx\; A(K). \tag{66}$$

In particular, if $\theta \leq 1$

$$|A_r^n\{F_H\} - A_r^n\{F_{H+\theta}\}| < \frac{4}{r}\sqrt{\frac{2}{\pi}}\int_H^\infty e^{-\frac{x^2}{2r^2}}\,dx\; A(H+1).$$

Hence if $K - H < m$, m being an integer, we have

$$|A_r^n\{F_H\} - A_r^n\{F_K\}| < \frac{4}{r}\sqrt{\frac{2}{\pi}}\sum_1^m \int_{H+i}^\infty e^{-\frac{x^2}{2r^2}}\,dx\; A(H+i+1). \tag{67}$$

It follows that if

$$\lim_{p\to\infty} \frac{A(p+1)\displaystyle\int_p^\infty e^{-\frac{x^2}{2r^2}}\,dx}{A(p)\displaystyle\int_{p-1}^\infty e^{-\frac{x^2}{2r^2}}\,dx} < 1, \tag{68}$$

then

$$\lim_{H\to\infty} |A_r^n\{F_H\} - A_r^n\{F_K\}| = 0 \tag{69}$$

uniformly in n and K. This will be the case, for example, if $A(p)$ is a polynomial in p, or is of the form a^p. Under these circumstances

$$\begin{aligned} \lim_{n\to\infty} A_r^n\{F\} &= \lim_{n\to\infty}\lim_{H\to\infty} A_r^n\{F_H\} \\ &= \lim_{H\to\infty}\lim_{n\to\infty} A_r^n\{F_H\} = \lim_{H\to\infty} A_r\{F_H\} \end{aligned} \tag{70}$$

exists. We shall call this quantity $A_r\{F\}$. There is no difficulty in showing that formula (63) holds in this case also.

Now let us have an $F|f|$ of the form

$$F|f| = a_0 + \int_0^1 K_1(t)f(t)dt + \dots$$
$$+ \int_0^1 dt_1 \dots \int_0^1 dt_n K_n(t_1, \dots, t_n) f(t_1) \dots f(t_n) + \dots, \tag{71}$$

given that

$$A(u) = a_0 + u \int_0^1 |K_1(t)| dt + \dots$$
$$+ u^n \int_0^1 dt_1 \dots \int_0^1 dt_n |K_n(t_1, \dots, t_n)| + \dots \tag{72}$$

exists for all u and satisfies (68). Then if

$$\max f|(t)| \leq B,$$

we have as an obvious result that (1) at the beginning of this section is satisfied. Moreover

$$|F|f| - F|g|| \leq \left| \sum_1^\infty \int_0^1 dt_1 \dots \right.$$
$$\int_0^1 dt_n K_n(t_1, \dots, t_n) [f(t_1) \dots f(t_n) - g(t_1) \dots g(t_n)]$$
$$\leq \sum_1^\infty \int_0^1 dt_1 \dots \int_0^1 dt_n |K_n(t_1, \dots, t_n)|$$
$$\times \{ [\max|G(x)| + \max|f(x) - g(x)|]^n - [\max|g(x)|]^n \}. \tag{73}$$

If $\max|f(x)| < B$, and $\max|f(x) - g(x)| < \epsilon$,

$$|F|f| - F|g|| \leq \sum_1^\infty [(B+\epsilon)^n - B^n] \int_0^1 dt_1 \dots \int_0^1 dt_n |K_n(t_1, \dots, t_n)|$$
$$= \sum_1^\infty \int_B^{B+\epsilon} ny^{n-1} \int_0^1 dt_1 \dots \int_0^1 dt_n |K_n(t_1, \dots, t_n)|$$
$$\leq \sum_1^\infty \epsilon n (B+\epsilon)^{n-1} \int_0^1 dt_1 \dots \int_0^1 dt_n |K_n(t_1, \dots, t_n)|. \tag{74}$$

Series (74) converges for all ϵ, since series (72) has a convergent derivative series. Hence condition (2) may be proved to be satisfied. In other words, $F|f|$ has an average in the sense of this paper.

This result may be much generalized without any great difficulty. It may be extended to functionals containing Stieltjes integrals such as

$$a_0 + \int_0^1 f(t) dQ_1(t) + \dots$$
$$+ \int_0^1 \dots \int_0^1 f(t_1) \dots f(t_n) \, d^n Q_n(t_1, \dots, t_n) + \dots, \tag{75}$$

provided

$$A(u) = a_0 + \sum_1^\infty u^n \int_0^1 \dots \int_0^1 |d^n Q(t_1, \dots t_n)| \;^{16} \tag{76}$$

exists for all u and satisfies (68). This class of functionals includes such expressions as

$$\int_0^1 \int_0^1 \int_0^1 [f(t_1)]^m [f(t_2)]^n [f(t_3)]^p K(t_1, t_2, t_3) \, dt_1 \, dt_2 \, dt_3.$$

[16] Cf. Daniell, *Functions of Limited Variation in an Infinite Number of Dimension*, Annals of Mathematics, Series 2, Vol. 21, pp. 30–38.

On the other hand, $A(u)$ in (72) may be replaced by

$$a_0+\sqrt{u}\sqrt{\int_0^1 [K_1(t)]^2\,dt} + \ldots + u^{\frac{n}{2}}\sqrt{\int_0^1 \cdots \int_0^1 dt_1 \ldots dt_n\,[K_n(t_1, \ldots, t_n)]^2}, \tag{77}$$

the K's being all summable and of summable square, and by the use of the Schwarz inequality, (1) and (2) may be deduced. Again, we may under the proper conditions concerning ϕ show that

$$\phi(F_1|f|, \ldots, F_n|f|) \tag{78}$$

has an average in the sense of this paper, if $F_1, \ldots, F_n$ are such functionals as we have already described.

I here wish to discuss only such a functional as

$$F|f| = \int_0^1 \cdots \int_0^1 [f(t_1)]^{\mu_1} \ldots [f(t_\nu)]^{\mu_\nu} K(t_1, \ldots t_\nu)\,dt \ldots dt_\nu.$$

The average of this functional will be the limit as n increases indefinitely of

$$\sum_{k_1=1}^{n} \cdots \sum_{k_\nu=1}^{n} (2\pi)^{-\frac{n}{2}} r^{-n} n^{-\frac{n}{2}} \int_{-\infty}^{\infty} dx_1 \ldots \int_{-\infty}^{\infty} dx_n\, e^{-\frac{(x_1^2+\cdots+x_n^2)n}{2r^2}}$$
$$(x_1+\ldots+x_{k_1})^{\mu_1} \ldots (x_1+\ldots+x_{k_\nu})^{\mu_\nu}$$
$$\int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}} dt_1 \ldots \int_{\frac{k_\nu-1}{n}}^{\frac{k_\nu}{n}} dt_\nu\, K(t_1, \ldots, t_\nu)$$

$$= \sum_{k_1=1}^{n} \cdots \sum_{k_\nu=1}^{n} \frac{(2\pi)^{-\frac{\nu}{2}} r^{-\nu} n^{\frac{\nu}{2}}}{[k_1(k_2-k_1)\ldots(k_\nu-k_{\nu-1})]^{\frac{1}{2}}}$$
$$\int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_\nu\, e^{-\frac{\xi_1^2 n}{2r^2 k_1} - \cdots \frac{\xi_\nu^2 n}{2r^2(k_\nu-k_{\nu-1})^{\mu_1}}}$$
$$\xi_1(\xi_1+\xi_2)^{\mu_2} \ldots (\xi_1+\ldots+\xi_\nu)^{\mu_2}$$
$$\int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}} dt_1 \ldots \int_{\frac{k_\nu-1}{n}}^{\frac{k_\nu}{n}} dt_n\, K(t_1, \ldots, t_\nu), \tag{80}$$

because of (63). Expression (80) clearly approaches the limit

$$\int_0^1 dt_1 \ldots \int_0^1 dt_n\, K(t_1, \ldots, t_n) \frac{(2\pi)^{-\frac{\nu}{2}} r^{-\nu}}{[t_1(t_2-t_1) \ldots (t_\nu-t_{\nu-1})]^{\frac{1}{2}}}$$
$$\int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_\nu\, exp\left(\frac{-\xi_1^2}{2r^2 t_1} - \cdots - \frac{\xi_\nu^2}{2r^2(t_\nu-t_{\nu-1})}\right)$$
$$\xi_1^{\mu_1}(\xi_1+\xi_2)^{\mu_2} \ldots (\xi_1+\ldots+\xi_\nu)^{\mu_\nu}, \tag{81}$$

provided that K is of limited total variation. This agrees with a definition already obtained by the author.[17]

§10. **The Average of a Functional as a Daniell Integral.** Daniell[18] has discussed a generalized definition of an integral in the following manner: he starts with a set of functions $f(p)$ of general elements p. He assumes a class T_0 of such functions which is closed with respect to the operations, multiplication by a constant, addition, and taking the modulus. He also assumes that to each f of class T_0 there corresponds a number K, independent of p, such that

$$|f(p)| \leq K,$$

and that to each f there corresponds a finite "integral" $U(f)$ having the properties

(C) $\quad Uc(f) = c\,U(f)$,

(A) $\quad U(f_1+f_2) = U(f_1) + U(f_2)$,

(L) $\quad$ If $\quad f_1 \geq f_2 \geq \ldots \geq 0 = \lim f_n$, then $\lim U(f_n) = 0$,

(P) $\quad U(f) \geq 0$ if $f \geq 0$.

There is no difficulty in showing that if T_0 be taken as the set of all functionals F such as those defined by (1) and (2) of §8, and the operator A_r is taken as U, all these conditions are ful-

[17] *The Average of an Analytic Functional*, p. 256.
[18] Daniell, p. 280.

filled save possibly (L). I here wish to discuss the question as to whether (L) is fulfilled.

Let it be noted that (L) involves the knowledge of those entities which form the arguments to the members of T_0. Up to this point we have regarded the arguments of our functionals F as step-functions. However, if $F|f|$ is a bounded, uniformly continuous functional of all step-functions constant over every interval $\left(\frac{k-1}{n},\frac{k}{n}\right)$, it may readily be shown that if f is the uniform limit of a sequence of such step-functions f_n, then $\lim F|f_n|$ exists, is independent of the particular sequence f_n chosen, and is bounded and uniformly continuous. We shall call this limit $F|f|$. This extension of the arguments of F does not vitiate the validity of any of the Daniell conditions. In fact, no condition save possibly (L) is vitiated if we then restrict the arguments of our functionals of T_0 to continuous functions $f(t)$ such that $f(0)=0$.

We wish, then, to show that if $F_1 \geq F_2 \geq \ldots \geq 0 = \lim F_n$, for every f that is continuous, then

$$\lim A_r\{F_n\}=0.$$

Let us notice that it follows from (50), (55) and (63) that

$$A_r\{F_n\} \leq [\max F_1]\frac{2}{r^2\pi}\sum_1^\infty n^2\, e^{-\frac{n^\epsilon ar}{4}} + \max_{S_a} F_n, \tag{82}$$

where S_a is the set of all functions f for which for every t_1 and t_2 between 0 and 1,

$$|f(t_2)-f(t_1)| \leq ar\,(t_2-t_1)^{1/2-\epsilon}.$$

By (57), the first term in (82) can be made as small as we please by taking a large enough. Accordingly, we can prove (L) if we can show that for every a, the set of functions Σ_n consisting in all the functions f in S_a for which $F_n|f| \geq \eta$ can be made to become null by making n large enough, whatever η may be. We shall prove this by a reductio ad absurdum.

Σ_n is in the language of Fréchet an extremal set, in that it is closed, equicontinuous, and uniformly bounded.[19] Hence either all the Σ_n's from a certain stage are null, or there is an element common to every Σ_n.[20] Let this element be $f(t)$. Then for all n, we have

$$F_n|f| \geq \eta \tag{83}$$

This however contradicts our hypothesis.

Our operation of taking the average is hence a Daniell integration, and as such capable of the extensions which Daniell develops. Daniell first proves that if $f_1 \leq f_2 \leq \ldots$ is a sequence from T_0, then the sequence $U(f_n)$ is an increasing sequence, and hence either becomes positively infinite or has a limit. If then $f=\lim f_n$ exists, Daniell defines $U(f)$ as $\lim U(f_n)$, and says that f belongs to T_1. Our T_1 will contain all the functions discussed in §9, and it admits of an easy proof that the definition of $A_r\{F\}$ in §9 accords with the definition arising from the Daniel extension of A_r whenever the former definition is applicable. Daniell then defines for any function f the upper semi-integral $\dot{U}(f)$ as the lower bound of $U(g)$ for g in T_1 and $g>f$. He defines $\underset{\cdot}{U}(f)$ as $-\dot{U}(-f)$, and $U(f)$ as $\dot{U}(f)$ if $\dot{U}(f)=\underset{\cdot}{U}(f)=$ finite, f being then called summable. All these extensions are applicable to our average operator A_r, as is also Daniell's theorem to the effect that if $f_1, \ldots, f_n, \ldots$ is a sequence of summable functions with limit f, and if a summable function ϕ exists such that $|f_n| \leq \phi$ for all n, f is summable, $\lim U(f_n)$ exists and $= U(f)$.

It may be shown from theorems of Daniell that if the measure of a set of functions be held to be the average of a functional 1 over the set and zero elsewhere, and the outer measure the upper semi-average of a functional 1 over the set and zero elsewhere, then the definition in §6 will coincide with this whenever it is applicable. §4 may be interpreted as saying that the measure of the set of functions for which the non-differentiability coefficient differs from r by more than ϵ is zero, and §7 as saying that the outer measure of the functions satisfying (51) is less than (55).

[19] *Sur quelques points du calcul fonctionnel*, Rend. Cir. Math. di Palermo Vol. 22, pp. 7, 37.

[20] Ibid, p. 7.

§11. **Independent Linear Functionals.** Let us consider a functional

$$F|f| = \phi\left(\int_0^1 K(x)f(x)dx, \int_0^1 Q(x)f(x)dx\right),$$

K and Q being summable of summable square, and ϕ being bounded and uniformly continuous for all arguments from $-\infty$ to ∞. We then have

$$F(x_1, \ldots, x_n) = \phi\left(\sum_1^n x_k \int_{\frac{k-1}{n}}^1 K(x)dx, \sum_1^n x_k \int_{\frac{k-1}{n}}^1 Q(x)dx\right).$$

$$= \phi\left\{\sum_1^n x_k \int_{\frac{k-1}{n}}^1 K(x)dx, \frac{\sum_1^n \int_{\frac{k-1}{n}}^1 K(x)dx \int_{\frac{k-1}{n}}^1 Q(x)dx}{\sum_1^n \left[\int_{\frac{k-1}{n}}^1 K(x)dx\right]^2} \times \sum_1^n x_k \int_{\frac{k-1}{n}}^1 K(x)dx \right.$$

$$\left. + \sum_1^n x_k \left[\int_{\frac{k-1}{n}}^1 Q(x)dx - \frac{\sum_1^n \int_{\frac{j-1}{n}}^1 K(x)dx \int_{\frac{j-1}{n}}^1 Q(x)dx}{\sum_1^n \left[\int_{\frac{j-1}{n}}^1 K(x)dx\right]^2} \times \int_{\frac{k-1}{n}}^1 K(x)dx\right]\right\}$$

$$= \phi\left\{\xi \sqrt{\sum_1^n \left[\int_{\frac{k-1}{n}}^1 K(x)dx\right]^2}, \ \xi \frac{\sum_1^n \int_{\frac{k-1}{n}}^1 K(x)dx \int_{\frac{k-1}{n}}^1 Q(x)dx}{\sqrt{\sum_1^n \left[\int_{\frac{k-1}{n}}^1 K(x)dx\right]^2}} \right.$$

$$\left. + \eta \sqrt{\sum_1^n \left[\int_{\frac{k-1}{n}}^1 Q(x)dx\right]^2 - \frac{\left[\sum_1^n \int_{\frac{k-1}{n}}^1 K(x)dx \int_{\frac{k-1}{n}}^1 Q(x)dx\right]^2}{\sum_1^n \left[\int_{\frac{k-1}{n}}^1 K(x)dx\right]^2}}\right\},$$

ξ and η being two orthogonal unit functions. As η increases, the arguments of ϕ approach uniformly

$$\xi \sqrt{\int_0^1 \left[\int_x^1 K(x)dx\right]^2 dx}$$

and

$$\xi \frac{\int_0^1 \left[\int_x^1 K(x)dx \int_x^1 Q(x)dx\right] dx}{\sqrt{\int_0^1 \left[\int_x^1 K(x)dx\right]^2 dx}}$$

$$+ \eta \sqrt{\int_0^1 \left[\int_x^1 Q(x)dx\right]^2 dx - \frac{\left[\int_0^1 \left[\int_x^1 K(x)dx \int_x^1 Q(x)dx\right] dx\right]^2}{\int_0^1 \left[\int_x^1 K(x)dx\right]^2 dx}}.$$

If in particular $\int_x^1 K(x)dx$ and $\int_x^1 Q(x)dx$ are normal and orthogonal,

$$A_r\{F\} = \lim_{n \to \infty} \frac{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1}\theta_1 \cos^{n-2}\theta_2 \, \phi(\sin\theta_1, \sin\theta_2) \, d\theta_1 \, d\theta_2}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1}\theta_1 \cos^{n-2}\theta_2 \, d\theta_1 \, d\theta_2}$$

$$= \frac{1}{2\pi r^2} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \, e^{-\left(\frac{x_1^2 + x_2^2}{2r}\right)} \phi(x_1, x_2). \qquad (84)$$

That is, the average of F with respect to f can be obtained by first forming the average with respect to f of

$$\phi\left(\int_0^1 K(x)f(x)dx,\ \int_0^1 Q(x)g(x)dx\right),$$

then forming the average of this average with respect to g, and finally putting $f=g$. A similar theorem can be shown by the same means to hold of

$$\phi\left(\int_0^1 K_1(x)f(x)dx,\ .\ .\ .\ .\ ,\ \int_0^1 K_n(x)f(x)dx\right),$$

in case the set of functions $\left[\int_x^1 K_k(x)dx\right]$ is normal and orthogonal.

If now ϕ is not a bounded function, but is merely uniformly continuous over any finite ranges of its arguments, and

$$\frac{1}{(2\pi r^2)^{\frac{n}{2}}}\int_{\infty}^{\infty} dx_1\ .\ .\ .\ \int_{-\infty}^{\infty} dx_n e^{-\frac{\sum_1^n x_k^2}{2r^2}}\phi(x_1,\ .\ .\ .\ .\ ,\ x_n) \quad (85)$$

exists, it may be proved by a simple limit argument that (85) represents the average of

$$\phi\left(\int_0^1 K_1(x)f(x)dx,\ .\ .\ .\ ,\ \int_0^1 K_n(x)f(x)dx\right).$$

§12. **Fourier Coefficients and the Average of a Functional.** The functions

$$-\sqrt{2}\ \sin n\pi x = \int_x^1 n\pi\sqrt{2}\ \cos n\pi x$$

form a normal and orthogonal set. Accordingly if we have a functional

$$\phi(a_1,\ .\ .\ .\ .\ ,\ a_n)$$

of the function

$$f(x) = a_0 + a_1\sqrt{2}\ \cos\sqrt{\pi x} +\ .\ .\ .\ + a_n\sqrt{2}\ \cos n\pi x +,\ .\ .\ .$$

its average will be

$$\frac{1}{(2\pi r^2)^{\frac{n}{2}}}\int_{-\infty}^{\infty} dx_1\ .\ .\ .\ \int_{-\infty}^{\infty} dx_n\, e^{-\frac{\sum_1^n x_k^2}{2r^2}}\phi\left(\frac{x_1}{\pi},\ .\ .\ .\ .\ ,\ \frac{x_n}{\pi n}\right) \quad (86)$$

provided this exists and ϕ is continuous, or even provided ϕ is a sum of step-functions. In particular if $F|f(x)|=1$ if $a_m^2+\ .\ .\ .\ +a_n^2+\ .\ .\ .\ \geq a^2$ and zero otherwise, its average will be less than the average of a functional which is 1 if for some k between m and n included,

$$a_k^2 > \frac{a^2 k^{-\frac{3}{2}}}{\sum_1^\infty j^{-\frac{3}{2}}}$$

and zero otherwise. The upper average of this latter functional is by (86) not greater than

$$\sum_m^\infty \frac{2}{(2\pi r^2)^{\frac{1}{2}}}\int_{\frac{ak^{\frac{1}{4}}\pi}{\left(\sum_1^\infty j^{-\frac{3}{2}}\right)^{\frac{1}{2}}}}^{\infty} e^{-\frac{x^2}{2}}dx$$

$$< \frac{2}{(2\pi r^2)^{\frac{1}{2}}}\sum_m^\infty L^{-k^{\frac{1}{4}}}, \quad (87)$$

for m sufficiently large, where

$$L = e^{\overline{\left(\sum_1^\infty j^{-\frac{3}{2}}\right)^{\frac{1}{2}}}^{\,a\pi}} > 1.$$

Series (87) converges. Hence as m increases, the measure of the set of functions for which $a_m^2+\ .\ .\ .\ +a_n^2+\ .\ .\ .\ > a^2$ approaches zero.

In this demonstration, we have made use of several theorems of Daniell which it did not seem worth while to enumerate in detail. They may all be found in his discussion of measure and integration.[21]

[21] Daniell, loc. cit.

Let us now consider a bounded functional $F|f|$ that is uniformly continuous in the more restrictive sense that for any positive η, there is a positive θ such that

$$|F|f|-F|g||<\eta$$

whenever

$$\int_0^1 [f(t)-g(t)]^2 dt<\theta.$$

Let $F|f|$ be invariant, moreover, when a constant is added to f. It will then be possible to write $F|f|$ in the form

$$F|f|=\phi(a_1,\ \ldots,\ a_n,\ \ldots),$$

where

$$|\phi(a_1\ \ldots,\ a_n,0,0\ \ldots)-\phi(a_1,\ \ldots,\ a_n,a_{n+1},\ \ldots)|<\eta$$

whenever

$$\sum_{n+1}^{\infty} a_k{}^2<\theta.$$

Hence

$$\left|A_r\{F\}-\frac{1}{(2\pi r^2)^{\frac{n}{2}}}\int_{-\infty}^{\infty}dx_1,\ \ldots\int_{-\infty}^{\infty}dx_n e^{-\frac{\sum_1^n x_k{}^2}{2r^2}}\right.$$

$$\left.\times\phi\left(\frac{x_1}{\pi},\ \ldots,\frac{x_n}{\pi_n},0,0,\ldots\right)\right|<\eta+2\max|F|M(\theta). \quad (87)$$

where $M(\theta)$ is the outer measure of all the functions for which $\sum_{n+1}^{\infty} a_k{}^2\geq\theta$. N can, as we have seen, be made arbitrarily small by making n sufficient y large. It can hence be shown that

$$A_r\{F\}=\lim_{n\to\infty}\frac{1}{(2\pi r^2)^{\frac{n}{2}}}\int_{-\infty}^{\infty}dx_1\ \ldots\int_{-\infty}^{\infty}dx_n e^{-\frac{\sum_1^n x_k{}^2}{2r^2}}$$

$$\times\phi\left(\frac{x_1}{\pi},\ \ldots,\ \frac{x_n}{\pi n},\ 0,0,\ \ldots\right). \quad (88)$$

This theorem may be established for the more general case of f functional F which is uniformly continuous for all functions a for which

$$\int_0^1 [f(x)]^2\,dx<\theta,$$

whatever θ may be, and for which there is an increasing $\psi(u)$ such that

$$F|f|\leq\psi\left[\int_0^1 [f(x)]^2 dx\right],$$

while

$$\sum_1^{\infty}\psi(n+1)[M(n+1)-M(n)]$$

converges.

Now let us consider a functional of the form

$$F|f|=H_{i_1}\left(\frac{x_1\pi}{r\sqrt{2}}\right)\ \ldots\ H_{i_n}\left(\frac{nx_n\pi}{r\sqrt{2}}\right), \quad (89)$$

where the H's are any set of Hermite polynomials corresponding to normalized Hermite functions. If $G|f|$ is another such function, it is easy to show that

$$A_r\{F|f|G|f|\}=0, \quad (90)$$

and to compute

$$A_r\{F|f|\}^2. \quad (91)$$

Now let us arrange all these functionals in a progression $\{F_n\}$ It follows from theorems analogous to those familiar in the ordinary theory of orthogonal functions that if $G|f|$ is a continuous function ϕ of $a_1,\ \ldots,\ a_n$ for which expression (86) exists, then

$$\sum_1^{\infty}F_k|f|\frac{A_r\{F_k|f|G|f|\}}{\sqrt{\{A_rF_k|f|\}^2}} \quad (92)$$

converges in the mean to $G|f|$ in the sense that if $G_n|f|$ is its nth partial sum,

$$\lim_{n\to\infty}A_r\{G|f|-G_n|f|\}^2=0. \quad (93)$$

With the aid of generalizations of familiar theorems concerning orthogonal functions, it may be shown that this result remains valid if $\{G|f|\}^2$ fulfils the conditions laid down for F in (88), and G does likewise. The functionals (89) are then in a certain sense a complete set of normal and orthogonal functions.

One final remark. By methods which exactly duplicate §4, it can be shown that the set of functions f for which it is not true that

$$\left|\frac{r^2}{\pi^2} - \lim_{n\to\infty}\sum_1^n \frac{k^2 a_k^2}{n}\right| < \epsilon \tag{94}$$

is of zero measure, whatever ϵ. This suggests interesting questions relating to the connection between the coefficient of non-differentiability of a function and $\lim_{n\to\infty}\sum_1^n \frac{k^2a_k^2}{n}$.